

# The production of a non-homogeneous classical pion field and the distribution of the neutral and charged pions

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## Abstract

The probability distribution  $dw/df$  as a function of the ratio  $f = n_0/n_{tot}$  of the neutral to total multiplicities is calculated for the classical pion fields quickly varying in space and time.

1. It is widely discussed in the recent years that the production of a classical pion field leads to the ratio

$$f = \frac{n_0}{n_{tot}} \quad (1)$$

of the multiplicity of the neutral pions to the total multiplicity which completely differs from what one expects for the usual mechanisms of the pion production. For the large values of  $n_{tot}$  the standard mechanisms predicts the probability

$$\frac{dw}{df} = \delta\left(f - \frac{1}{3}\right), \quad (2)$$

while for a particular classical state known as the "disoriented chiral condensate" (DCC), the distribution is

$$\frac{dw}{df} = \frac{1}{2\sqrt{f}} \quad (3)$$

(see, for example, [1] and a mini-review [2] and the references therein).

In this letter we shall discuss a class of classical chiral fields described in [3] and show that the distribution (3) is correct only for the fields slowly varying in space and time (for instance, for the DCC which is an exact constant) whereas for the fields which vary rapidly enough the predicted distribution is quite different (the plot of the latter distribution is shown in Fig.1).

The classical solutions of the non-linear  $\sigma$ -model found in [3] can be represented in the following form. As usually the four fields  $\sigma, \vec{\pi}$  are constrained by the relation:

$$\sigma^2 + \vec{\pi}^2 = f_\pi^2. \quad (4)$$

If we denote  $\Phi_a = f_\pi^{-1}(\vec{\pi}, \sigma)$ ,  $a = 1, 2, 3, 4$ , the solution is

$$\Phi_a = A_a \cos(\theta(x) + \varphi_a) , \quad (5)$$

where the amplitudes  $A_a$  and the phases  $\varphi_a$  obey the equations:

$$\sum_{a=1}^4 A_a^2 = 2 , \quad \sum_{a=1}^4 A_a^2 e^{2i\varphi_a} = 0 , \quad (6)$$

while the function  $\theta(x)$  satisfies the free wave equation:

$$\partial^2 \theta(x) = 0 . \quad (7)$$

The derivation and the discussion of these solutions see in ref.[3].

Let us introduce the complex-4-vector in the chiral  $O(4)$  space:

$$Z_a = A_a e^{i\varphi_a} = X_a + iY_a . \quad (8)$$

Eqs. (6) can be rewritten

$$X_a^2 = 1 , \quad Y_a^2 = 1 , \quad X_a Y_a = 0 . \quad (9)$$

(Here  $X_a^2 = \sum_{a=1}^4 X_a^2$ , etc.).

The  $O(4)$  invariant dynamics for the production of the states characterized by the parameters  $X_a, Y_a$  can lead to the distribution function which depends only on the invariants  $X_a^2, Y_a^2, X_a \cdot Y_a$ :

$$dw = \rho(X_a^2, Y_a^2, X_a \cdot Y_a) d^4 X d^4 Y . \quad (10)$$

Due to Eqs.(9) the function  $\rho$  is proportional to the  $\delta$ -functions:  $\delta(X_a^2 - 1)\delta(Y_a^2 - 1)\delta(X_a Y_a)$ , and since all the invariants turn out to be fixed (10) takes the form:

$$dw = \text{const} \delta(X_a^2 - 1)\delta(Y_a^2 - 1)\delta(X_a Y_a) d^4 X d^4 Y . \quad (11)$$

The number of the neutral pions  $n_0$  and the total number of the pions  $n_{tot}$  are the functions of  $X_a, Y_a$ . The distribution in the variable  $f$ , the fraction of neutral pions, is given by:

$$\frac{dw}{df} = \text{const} \int \delta\left(f - \frac{n_0(X, Y)}{n_{tot}(X, Y)}\right) \delta(X_a^2 - 1)\delta(Y_a^2 - 1)\delta(X_a Y_a) d^4 X d^4 Y . \quad (12)$$

The ratio  $n_0/n_{tot}$  is proportional to ratio of the squared amplitudes of the corresponding pion fields integrated over the production volume:

$$\frac{n_0(X, Y)}{n_{tot}(X, Y)} = \frac{\int d^3 r \Phi_3^2}{\int d^3 r [\Phi_1^2 + \Phi_2^2 + \Phi_3^2]} . \quad (13)$$

According to (5):

$$\Phi_i^2 = A_i^2 \cos^2 \left( \theta(x) + \varphi_i \right), \quad i = 1, 2, 3. \quad (14)$$

Two different limiting cases are possible.

If  $\theta(x)$  varies slowly through the production volume,  $\theta(x) \approx \text{const}$ , one can redefine the phases and put  $\theta = 0$ . Then

$$\begin{aligned} \Phi_i^2 &= A_i^2 \cos^2 \varphi_i = X_i^2, \\ \frac{n_0(X, Y)}{n_{tot}(X, Y)} &= \frac{X_3^2}{X_1^2 + X_2^2 + X_3^2}. \end{aligned} \quad (15)$$

This case of the constant field is actually the case of the DCC.

Substituting (15) into Eq.(12) we can easily perform an integration over  $d^4Y$  and the three of the four  $d^4X$  integrations to get:

$$\frac{dw}{df} = \text{const} \int_{-1}^{+1} d(\cos \theta) \delta(f - (\cos \theta)^2) = \text{const} \frac{1}{\sqrt{f}}. \quad (16)$$

Here

$$\cos \theta = \frac{X_3}{\sqrt{X_1^2 + X_2^2 + X_3^2}}. \quad (17)$$

Thus we see that the inverse square root law comes out when the field varies slowly through the production volume.

For the opposite case we write  $\cos^2(\theta(x) + \varphi_i) = 1/2(1 + \cos(2\theta(x) + 2\varphi_i))$  and assume that the term linear in  $\cos(2\theta + 2\varphi_i)$  is integrated out in Eq.(13). Then

$$\frac{n_0(X, Y)}{n_{tot}(X, Y)} = \frac{A^2}{\bar{A}^2} = \frac{X_3^2 + Y_3^2}{\bar{X}^2 + \bar{Y}^2}, \quad \bar{X}^2 = \sum_{i=1}^3 X_i^2, \quad \bar{Y}^2 = \sum_{i=1}^3 Y_i^2. \quad (18)$$

The integral, which appears when this expression is substituted into Eq.(12), is less trivial than the one for the previous case. In fact it appears possible to perform the 7 of the 8 integrations  $d^4X d^4Y$  analytically. This is described in the second part of this letter. A reader who is not interested in the technical details turn to the plot of  $J(f) = dw/df$  shown in Fig.1. Note also that

$$J(0) = \frac{4}{3}, \quad J\left(\frac{1}{2}\right) = -\frac{5}{2\sqrt{2}} \ln \left| \tan \frac{\pi}{8} \right| + \frac{1}{2} = 2.058, \quad J(1) = 0. \quad (19)$$

The discontinuity of  $dJ(f)/df$  at  $f = 1/2$  is elucidated qualitatively in the next section of the letter.

2. The properly normalized distribution which emerges from Eqs. (12) and (8) is

$$\begin{aligned} \frac{dw}{df} = J(f) &= \frac{1}{2\pi^3} \int d^4X d^4Y \delta \left( f - \frac{X_3^2 + Y_3^2}{\bar{X}^2 + \bar{Y}^2} \right) \delta(x_a^2 - 1) \delta(Y_a^2 - 1) \delta(X_a Y_a), \\ \int_0^1 J(f) df &= 1. \end{aligned} \quad (20)$$

We split  $d^4X = d^2x_\perp d^2x_\parallel$  and  $d^4Y = d^2y_\perp d^2y_\parallel$ , where  $x_\perp = (X_1, X_2)$ ,  $x_\parallel = (X_3, X_4)$ ,  $y_\perp = (Y_1, Y_2)$ ,  $y_\parallel = (Y_3, Y_4)$ , and perform the integrations  $d^2x_\perp d^2y_\perp$  using the three  $\delta$ -functions. One easily obtains:

$$J(f) = \frac{1}{2\pi^2} \int d^2x_\parallel dy_\parallel \delta\left(f - \frac{X_3^2 + y_3^2}{2 - X_4^2 - Y_4^2}\right) \times \\ \times \frac{\Theta[(1 - x_\parallel^2)(1 - y_\parallel^2) - (\vec{x}_\parallel \vec{y}_\parallel)^2] \Theta(1 - x_\parallel^2) \Theta(1 - y_\parallel^2)}{\sqrt{(1 - x_\parallel^2)(1 - y_\parallel^2) - (\vec{x}_\parallel \vec{y}_\parallel)^2}}, \quad (21)$$

where  $\Theta$ 's are the step-functions. The last two  $\Theta$  functions can be rewritten in the following way. To have  $1 - x_\parallel^2 > 0$  and  $1 - y_\parallel^2 > 0$  it is necessary and sufficient to have the product of these factors and their sum to be positive. The product  $(1 - x_\parallel^2)(1 - y_\parallel^2)$  is anyway positive due to the first  $\Theta$  function. Therefore we can change  $\Theta(1 - x_\parallel^2)\Theta(1 - y_\parallel^2) \rightarrow \Theta(2 - x_\parallel^2 - y_\parallel^2)$ .

Using Eq.(8) we rewrite the integrations  $d^2x_\parallel d^2y_\parallel = A_3 dA_3 d\varphi_3 A_4 dA_4 d\varphi_4$  and readily get for  $J(f)$ :

$$J(f) = \frac{4}{\pi} \int_0^{\pi/2} d\phi \int_0^1 d\lambda (1 - \lambda) \frac{\Theta[1 - 2\lambda - 2f(1 - \lambda) + 4\lambda(1 - \lambda)f \sin^2 \varphi]}{\sqrt{1 - 2\lambda - 2f(1 - \lambda) + 4f\lambda(1 - \lambda) \sin^2 \varphi}}. \quad (22)$$

In Eq.(22) we used  $\lambda = 1/2 A_4^2$  and  $\varphi = \varphi_3 - \varphi_4$ .

Two limiting cases are immediately obtained from (22). For  $f = 1$  the argument of the  $\Theta$  function is negative (except for one point:  $\lambda = 1/2$ ,  $\sin^2 \varphi = 1$ ) and therefore  $J(1) = 0$ . For  $f = 0$  one has:

$$J(0) = 2 \int_0^{1/2} \frac{d\lambda(1 - \lambda)}{\sqrt{1 - 2\lambda}} = \frac{4}{3}. \quad (23)$$

For the arbitrary  $f$  it is convenient to perform the  $\lambda$  integration to get:

$$J(f) = \frac{4}{\pi} \int_0^{\pi/2} d\varphi \left\{ \frac{\sqrt{a\lambda^2 + b\lambda + c}}{-a} - \frac{1 + b/2a}{\sqrt{-a}} \arcsin \frac{2a\lambda + b}{\sqrt{b^2 - 4ac}} \right\} \Bigg|_{\lambda=\lambda_{\min}, \text{ or } 0}^{\lambda=\lambda_{\max}, \text{ or } 1}. \quad (24)$$

Here

$$a = -4f \sin^2 \varphi, \quad b = -2(1 - f) + 4f \sin^2 \varphi, \quad c = 1 - 2f, \\ \lambda_{\max, \min} = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}. \quad (25)$$

The limits  $(\lambda_{\min}, \lambda_{\max})$  in Eq.(24) should be understood in the following way. If  $b^2 - 4ac < 0$  then  $J = 0$  (since the argument of the  $\Theta(a\lambda^2 + b\lambda + c)$  in Eq.(22) is negative). For  $b^2 - 4ac > 0$  the integration in  $\lambda$  is taken between  $\lambda_{\min}$  and  $\lambda_{\max}$  only if these quantities are inside the interval  $(0, 1)$ .

The last integration in (24) over  $\varphi$  has been performed numerically and led to the plot for  $J(f)$  depicted in Fig.1. We would like to make two comments concerning this plot.

First, one can analytically calculate  $J(f)$  for  $f = 1/2$ . From Eq.(24) one obtains:

$$J(1/2) = \sqrt{2} \int_{\pi/4}^{\pi/2} d\varphi \left[ \frac{1}{\sin \varphi} + \frac{1}{2 \sin^3 \varphi} \right] = -\frac{5}{2\sqrt{2}} \ln \tan \frac{\pi}{8} + \frac{1}{2} = 2.058 . \quad (26)$$

Second, to understand qualitatively the discontinuity of  $dJ(f)/df$  at  $f = 1/2$  one can notice that for the fixed value  $\varphi = \pi/4$  the integrand in Eq.(24) has a gap for  $f = 1/2 + \varepsilon$  when  $\varepsilon \rightarrow +0$ .

Indeed, for  $\varphi = \pi/4$  one has  $a = -1 - 2\varepsilon$ ,  $b = +4\varepsilon$ ,  $c = -2\varepsilon$ . Since it will be immediately evident that only small  $\lambda \sim \sqrt{|\varepsilon|}$  are essential the polynom  $a\lambda^2 + b\lambda + c \simeq -\lambda^2 - 2\varepsilon$ . The integral (22) in  $\lambda$  (i.e. the integrand in(24)) is then

$$\frac{4}{\pi} \int_0^1 d\lambda \frac{\Theta(-2\varepsilon - \lambda^2)}{\sqrt{-2\varepsilon - \lambda^2}} = \begin{cases} 4 & \text{for } \varepsilon < 0 \\ 0 & \text{for } \varepsilon > 0 . \end{cases} \quad (27)$$

The integration over  $\varphi$  (within the interval  $\Delta \varphi \sim \sqrt{\varepsilon}$ , where  $b\lambda \leq |c|$ ) smears out the discontinuity in  $J(f)$  but the derivative  $dJ/df$  still has "singularity"  $dJ/df \sim -\Theta(\varepsilon)/\sqrt{\varepsilon}$  at  $\varepsilon \rightarrow +0$ .

3. In ref.[2] we discussed the possible production mechanisms and the signatures which can be used for the observation of the classical pion field. One of the conclusions of this paper was that it is not easy to observe the DCC, or almost constant pion field, due to a simple reason: the energy density for such a state is so small that no more than tenths of the pions can be produced through the DCC decay. We claimed that the observation of the field varying in space and time may be more promising since in this case the energy density can be much higher. We see now that the distribution in  $f = n_0/n_{tot}$  for such fields may be quite different from what is usually supposed, i.e. from  $dw/df \sim 1/\sqrt{f}$ , and that for a class of the fields described in [3]  $dw/df$  is given by Eq.(24) (Fig.1). It also seems that the latter fields are closer to those discussed in [4] than the constant DCC.<sup>1</sup>

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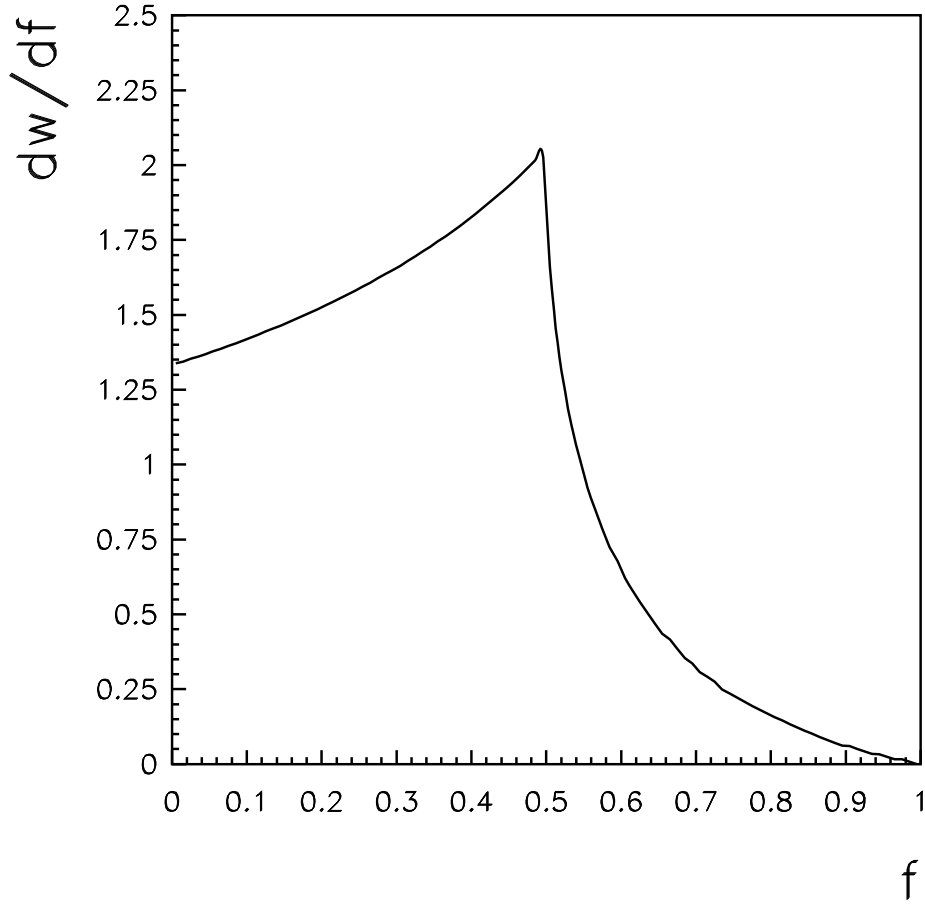


Figure 1: The probability distribution  $dw/df$  over the neutral to total multiplicities ratio  $f = n_0/n_{tot}$  for classical pion fields quickly varying in space and time.